Problem:

The correlation function $K_X(\tau)$ of the stationary random function $X(t)$ is given: $K_X(\tau)=\sigma^2 e^{-\alpha^2\tau^2}$.

Find the correlation function of the random function $Y(t)=aX'(t)$.

Solution:

$K_X(\tau)=\sigma^2 e^{-\alpha^2\tau^2}$, $Y(t)=aX'(t)$.

We have the theorem:

The correlation function of the derivative of the stationary random function is equal to the second derivative of its correlation function, taken with a minus sign:

$$K_{X'}(\tau)=-K_X''(\tau), \qquad K_X'(\tau)=\frac{\partial}{\partial\tau}\left(\sigma^2 e^{-\alpha^2\tau^2}\right)=\sigma^2\cdot(-2\alpha^2\cdot\tau)e^{-\alpha^2\tau^2},$$

$$K_X''(\tau)=\frac{\partial}{\partial\tau}\left(-2\sigma^2\alpha^2\tau\cdot e^{-\alpha^2\tau^2}\right)=-2\alpha^2\sigma^2\left(e^{-\alpha^2\tau^2}+\tau(-2\alpha^2\tau)e^{-\alpha^2\tau^2}\right)=-2\alpha^2\tau^2\cdot(1-2\alpha^2\tau^2)\cdot e^{-\alpha^2\tau^2},$$

$$K_Y(\tau)=K_{aX'}(\tau)=a^2K_{X'}(\tau)\Rightarrow K_Y(\tau)=-a^2K_X''(\tau),\Rightarrow \boxed{K_Y(\tau)=2a^2\alpha^2\tau^2(1-2\alpha^2\tau^2)e^{-\alpha^2\tau^2}}.$$